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ROBERT STEPHENSON, M.P., President, in the Chair

Mr. ROBERT STEPHENSON, PRESIDENT:- "Gentlemen, before calling on our friend Mr. Bidder to commence his lecture, I need scarcely offer an apology for thus sanctioning a deviation from the usual course, in favour of receiving an oral address to this Institution; the present case is so exceptional, that the Council unite with me in appreciating the difficulty experienced by Mr. Bidder in endeavouring to commit to paper his ideas on the abstruse subject of mental calculation; indeed, so great was that difficulty, that the subject must have been abandoned, if the Council had not consented to receive it in the form of an oral address.

It will require great dexterity on the part of the speaker, to elucidate all the processes, and will demand your careful attention to comprehend, and appreciate the power and ability necessary to be brought to bear on the subject, to the practical application of which his early years were devoted, and which, after a long interval, he has now resumed, with the intention of endeavouring to render intelligible those very difficult processes of the mind. There have been several instances of the possession of great power of mental calculation, and of extreme rapidity of multiplying figures and tracing their mutual relation, but I believe, that hitherto, none of these gifted persons, even after enjoying the benefits of education, have been enabled to render intelligible the source of the power, or to describe clearly the processes employed.

Even after his exposition, it is probable that Mr. Bidder may not be able to make us all mental calculators; but I have little doubt of his succeeding in explaining clearly to us, some of those wonderful properties of his own mind which have rendered him so conspicuous among those few persons in the world, who have possessed that extraordinary power of dealing with figures; I therefore ask your careful attention to the remarks of our friend."

"On Mental Calculation"
by George Parker Bidder, Vice-President.

Gentlemen, my very kind and worthy friend, the President, has explained to you the circumstances which render it necessary, that the experience I have had as a mental calculator, and the views I entertain on the subject of mental calculation, should be communicated verbally to the meeting, instead of being transmitted to you through our worthy Secretary. But, before I commence my remarks, I must solicit your kindest indulgence and your greatest forbearance. I am about to undertake a task to which I am totally unaccustomed, that of offering to a public assembly an address of the length to which I feel this subject will extend. I trust, however, that if anything be left obscure, or unexplained, you will make a note of the points requiring elucidation; and before the discussion is closed I will endeavour to afford the beat explanation in my power, and to supply that which may have been left incomplete.

I have frequently attempted to reduce to writing my views on the subject, and if all I had to communicate to you were merely a, short description, or an abbreviated process, I could easily commit those views to paper. But the points to which I am anxious to direct your attention are the principal operations which are concerned in mental computation. I desire, as far as I can, to lay open my mind to you, and to exhibit the rapid evolutions which it, undergoes in mental computation. It is not for me to say, that this task is beyond the power of those who are accustomed to writing, but it is certainly far beyond any powers that I possess.

I have, for many years, entertained a strong conviction, that mental arithmetic can be taught, as easily, if not even with greater facility, than ordinary arithmetic, and that it may be rendered conducive to more useful purposes, than that of teaching by rule; that it may be taught in such a way as to strengthen the reasoning powers of the youthful mind; so to enlarge it, as to ennoble it and to render it, capable of embracing all knowledge, particularly that appertaining to the exact sciences. Whether this view be right, or wrong, since I have entertained it, I have felt it was due to society to communicate my impression; leaving to others, should there be any practical utility in the suggestion, the task of carrying it into effect. My time has, moreover, been for many years wholly devoted to professional avocations, and I have accordingly, with that disposition to procrastinate, which more, or less belongs to us all, delayed the performance of this duty from time to time; - and I might perhaps still have delayed it, but that I was determined by a strong desire to seize the first opportunity afforded, during the occupation of the Chair by my oldest and best friend.

I must, in the outset, mention that those friends to whom I have, at various times, communicated my view, that mental arithmetic could be taught, have almost invariably dissented from me. They have urged, that although there are several instances on record, in which the faculty of calculation has been remarkably developed, yet, comparatively speaking, - and in comparison with the masses of the population in the world, those instances have been very few and very rare. In fact, Jedediah Buxton and Zerah Colborne are the only two mental calculators who have attained any great degree of celebrity in modern days. It has been further urged that, in order to attain eminence in the science of mental arithmetic, there must be an especial turn of mind, an extraordinary power of memory, and great mathematical aptitude. I have endeavoured to examine my own mind, to compare it with that of others, and to discover if such be the case, but I can detect no particular turn of mind, beyond a predilection for figures, which many possess almost in an equal degree with myself. I do not mean to assert, that all minds are alike constituted to succeed in mental computation; but I do say that, as far as I can judge, there may be as large a number of successful mental calculators as there are - who attain eminence in any other branch of learning. As regards memory, I had in boyhood, at school and at college, many opportunities of comparing my powers of memory with those of others, and I am convinced that I do not possess that faculty in any remarkable degree.

If, however, I have not any extraordinary amount of memory, I admit that my mind has received a degree of cultivation in dealing with figures, in a particular manner, which has induced in it a peculiar power; I repeat, however, that this power is, I believe, capable of being attained by any one disposed to devote to it the necessary time and attention. In other respects than numbers, I have not an extraordinary memory; indeed, I have great difficulty in learning anything by rote. I may learn a page of literature, or poetry, but it is no sooner learned than it is forgotten. On the other hand, facts which I have been at some pains to obtain, and which have induced conviction after examination and reasoning upon them, when once fixed in my mind become indelible. In this, however, there is nothing extraordinary, and the majority of students have experienced the difference between learning lessons by heart and having them impressed on the mind by reasoning, explanation, or experiment. As regards mathematical aptitude, I cannot be mistaken, for when I was associated with large classes, I experienced considerable difficulty in maintaining a fair position, and in no respect have I ever been distinguished for mathematical pursuits; indeed, up to the present time I have no great fondness for mathematical formulae, particularly if they are very abstruse and repulsive in appearance.

I fear you may think that I am occupying your time uselessly with these observations, but I feel that I shall base my argument upon a strong position, if I can demonstrate to you, in the outset, that the exercise of mental calculation requires no extraordinary power of memory, and that mental arithmetic can be taught. I do not, however, mean to say that it ought to be taught, or that it is desirable to attempt to teach it, to the extent to which I have been enabled to carry it. I have sacrificed years of labour, - I have striven with much perseverance, to obtain, and to retain a power, or mastery over numbers which will probably, at all times, be as rare, as its utility in the ordinary affairs of life.

Far be it, from me, however, to say that it has been of little use to me. Undoubtedly the acquirement has attracted towards me, a degree of notice, which has ended in raising me from the position of a common labourer, in which I was born, to that of being able to address you as one of the Vice-Presidents of this distinguished Society. But as I have already said, I am not about to lay before you any abbreviated process of calculation; there are no "royal roads" to mental arithmetic. Whoever wishes to achieve proficiency in that, as in any other branch of science, will only succeed by years of labour and of patient application. In short, in the solution of any arithmetical question, however simple, or complicated, every mental process must be analogous to that which is indicated in working out algebraical formula. No one step can be omitted; but all and every one must be taken up one after another, in such consecutive order, that if reduced to paper, the process might appear prolix, complicated, and inexpeditious, although it is actually arranged with a view of affording relief to the memory. And here let me say, that the exercise of the memory is the only real strain on the mind, and which limits the extent to which mental calculation may be carried. It may be imagined that this is somewhat inconsistent with my previous observation that I possess no extraordinary power of memory. But it must be borne in mind, that my memory is the limit by which my mental powers are restricted; and that the processes I pursue are all adopted, simply with a view of relieving the registering powers of the mind, i. e., the memory.

Now, taking you back to your early infancy, endeavour to recall the first things to which your attention was invited. As infants you were first taught to speak; - you were then taught letters; - then the combination of letters into words; then of words into sentences; and after that you gradually acquired an extensive vocabulary of words and facts. We possess and store these words and facts in our minds, to be occasionally called forth as we need them. For instance, in reading the page of a book, it is clear to me, however rapidly you may read it, that every letter of that page passes in review through the mind. The mind first combines the letters upon the page into words, then the words into sentences, and, from those sentences, it extracts the meaning.

Now, in mental calculation I have accumulated, not a very great number of facts, after all; - but I do possess them, and although at this moment I am unconscious of their being so stored up, yet the moment I have a question to solve I have them instantly at command. And it appears to me that, in both cases, the phenomena may be compared to that which we have all observed in Nature. If, on a dark night, there occurs a storm of lightning, during the instant of the flash, although immeasurable in point of time, every object is rendered clear, and out of that view, so placed before us, we can select some one object for our consideration. So I believe it is in the mind; whenever, as in calculation, I feel called upon to make use of the stores of my mind, they seem to rise with the rapidity of lightning. The reasoning faculty seizes upon a particular series of facts necessary for the purpose, deals with each fact according as the circumstances require, and transmits it to the memory for registration. But the registration required for figures is very different from that demanded for ordinary occurrences.

An author, when writing on any subject, first forms the argument in his mind, or frames an outline of the plot, which he proposes to fill up; but in the mode of recording his views in writing, he has the advantage of an infinite variety of combinations of words, more or less clear and expressive, without feeling restricted to any particular words, or form of expression. But in mental computation there is no such latitude; if you wish to commit arithmetical processes to paper, you must record them in the exact form in which you have reasoned on them, and in their exact sequence and order; a wrong figure, or a figure misplaced would vitiate the whole result, and hence the great strain on the mind occasioned by mental computation; everything must be remembered with perfect accuracy, and when the number of impressions to be retained in the mind is large, the retention of them with sufficient distinctness is a work of great mental labour. Hence it is that where the impressions required are few and simple, they are taken up with great rapidity; but in proportion as the numbers increase, so the registration by the mind becomes more and more difficult, until at last the process becomes as slow as registration upon paper. When that point is arrived at, it is clear that the utility of mental calculation ceases, and the process ought to be carried on upon paper. But up to that point the velocity of the mental process cannot be adequately expressed; the utterance of words cannot equal it; in fact, as compared with the process of speaking, or of writing, it is as the velocity of a message transmitted by telegraph to the speed of an express train.

I can perhaps convey to you no stronger view of this subject than by mentioning, that, were my powers of registration at all equal to the powers of reasoning, or execution, I should have no difficulty, in an inconceivable short space of time, in composing a voluminous table of logarithms; but the power of registration limits the power of calculation, and as I said before, it is only with great labour and stress of mind that mental calculation can be carried on beyond a certain extent. Now, for instance, suppose that I had to multiply 89 by 73, I should say instantly 6497; if I read the figures written out before me I could not express a result more correctly, or more rapidly; this facility has, however, tended to deceive me, for I fancied that I possessed a multiplication table up to 100 times 100, and, when in full practice, even beyond that; but, I was in error; the fact is that I go through the entire operation of the computation in that short interval of time which it takes me to announce the result to you. I multiply 80 by 70, 80 by 3; 9 by 70, and 9 by 3; which will be the whole of the process as expressed algebraically, and then I add them up in what appears to be merely an instant of time.

This is done without labour to the mind; and I can do any quantity of the same sort of calculation without, any labour; and can continue it for a long period; but when the number of figures increases, the strain on the mind is augmented in a very rapid ratio. As compared with the operation on paper, in multiplying 3 figures by 3 figures, you have three lines of 4 figures each, or 12 figures in the process to be added up; in multiplying 6 figures into 6 figures, you have six lines of 7 figures, or 42 figures to be added up. The time, therefore, in registration on paper will be as 12 to 42. But the process in the mind is different. Not only have I that additional number of facts to create, but they must be imprinted on the mind. The impressions to be made are more in number, they are also more varied, and the impression required is so much deeper, that instead of being like 3 or 4 to 1, it is something like 10 to 1. Instead of increasing by the square, I believe it increases by the fourth power. I do not pretend to say that it can be expressed mathematically, but the ratio increases so rapidly that it soon limits the useful effect of mental calculation. As a great effort I have multiplied 12 places of figures by 12 places of figures; but that has required much time, and was a great strain upon the mind. Therefore, in stating my conviction that mental arithmetic could be taught, I would desire it to be understood, that the limits within which it may be usefully and properly applied, should be restricted to multiplying 3 figures by 3 figures. Up to that extent, I believe it may be taught with considerable facility, and will be received by young minds, so disposed, quite as easily as the ordinary rules of arithmetic.

The reason for my obtaining the peculiar power of dealing with numbers may be attributed to the fact, that I understood the value of numbers before I knew the symbolical figures. I learned to calculate before I could read, and therefore long before I knew one figure from another. In consequence of this, the numbers have, always had a significance and a meaning to me very different to that which figures convey to children in general. If a boy is desired to multiply 3487 by 3273 he goes through a certain process, which he has been taught dogmatically; he cannot explain the process, or the reasons for adopting it, but he arrives, almost mechanically, at the amount, - 11,412,951; which he has been taught is the result he should obtain, without any appreciation of what the figures represent, or how he arrives at them.

The process may, without exaggeration, be compared to the task of committing to memory a page of letters, instead of a page of words. Most of us would, without much difficulty, undertake to learn by heart a page of either prose, or poetry, but there are few among us who would undertake the same task with a page of letters. In fact there would be just the difference between attempting to remember a telegraphic message, transmitted by a code of arbitrary signals, and one sent in plain words.

After these necessary preliminary observations, I shall proceed to address myself more closely to the subject proposed to be treated, and in doing so, instead of submitting to you any speculation of my own, I shall rather endeavour to trace chronologically my own experience as a mental computator, commencing with my earliest years; I desire frankly to lay before you the steps I pursued in attaining the power of calculation, and to submit to you my own inferences, leaving you of course to draw your own conclusions, if they differ from those which I shall have the honour of offering to you. I propose therefore, with your kind permission, first to take you through the process of multiplication; and I begin with that rule because it is the basis of all calculations. Whoever has attained the power of multiplying with facility, either on paper, or mentally, possesses the elementary machine, which enables him to apply that power to any purpose of calculation. If a simile may be permitted, I would compare it to the case of a man who possesses a powerful steam engine, and whether he uses it for knitting stockings, or for pumping water from the depths of the bowels of the earth, the power is equally utilized; whilst without the machine, the limits of his individual force are soon reached.

As nearly as I can recollect, it was at about the age of six years, that I was first introduced to the science of figures. My Father was a working mason, and my elder Brother pursued the same calling. My first and only instructor in figures was that elder Brother, who was some years since removed from among us by death; the instruction he gave me was commenced by teaching me to count up to 10. Having accomplished this, he induced me to go on to 100, and there he stopped.

Having acquired a certain knowledge of numbers, by counting up to 100, I amused myself by repeating the process, and found, that by stopping at 10, and repeating that every time, I counted up to 100 much quicker than by going straight through the series. I counted up to 10, then to 10 again = 20, 3 times 10 = 30, 4 times 10 = 40, and so on. This may appear to you a simple process, but I attach the utmost importance to it, because it made me perfectly familiar with numbers up to 100; they became as it were my friends, and I knew all their relations and acquaintances. Yoy must bear in mind, that at this time I did not know one written, or printed figure from another, and my knowledge of language was so restricted, that I did not know there was such a word as "multiply;" but having acquired the power of counting up to 100 by 10 and by 5, I set about, in my own way, to acquire the multiplication table. This I arrived at by getting peas, or marbles, and at last I obtained a treasure in a small bag of shot: I used to arrange them into squares, of 8 on each side, and then on counting them throughout. I found that the whole number amounted to 64: by that process I satisfied my mind, not only as a matter of memory but as a matter of conviction, that 8 times 8 were 64; and that fact once established has remained there undisturbed until this day, and I dare say it will remain so to the end of my days. It was in this way that I acquired the whole multiplication table up to 10 times 10; beyond which I never went; it was all that I required.

At the period referred to, there resided, in a house opposite to my Father's, an aged Blacksmith, a kind old man who, not having any children, had taken a nephew as his apprentice. With this old gentleman I struck up an early acquaintance and was allowed the privilege of running about his workshop. As my strength increased, I was raised to the dignity of being permitted to blow the bellows for him, and on winter evenings I was allowed to perch myself on his forge hearth, listening to his stories. On one of these occasions, somebody by chance mentioned a sum, whether it was 9 times 9 or what it was I do not now recollect; but whatever it was, I gave the answer correctly. This occasioned some little astonishment; they then asked me other questions, which I answered with equal facility. They then went on to ask me up to two places of figures: 13 times 17 for instance; that was rather beyond me, at the time, but I had been accustomed to reason on figures, and I said 13 times 17 means 10 times 10 plus 10 times 7, plus 10 times 3 and 3 times 7. I said 10 times 10 are 100, 10 times 7 are 70, 10 times 3 are 30, and 3 times 7 are 21; which added together give the result, 221; of course I did not do it then as rapidly as afterwards, but I gave the answer correctly, as was verified by the old gentleman's nephew, who began chalking it up to see if I was right. As a natural consequence, this increased my fame still more, and what was better, it eventually caused halfpence to flow into my pocket; which I need not say, had the effect of attaching me still more to the science of arithmetic, and thus by degrees I got on, until the multiple arrived at thousands.

Then of course my powers of numeration had to be increased, and it, was explained to me that 10 hundreds meant 1000. Numeration beyond that point, is very simple in its features; 1000 rapidly gets up to 10,000 and 20,000, as it is simply 10, or 20 repeated over again, with thousands at the end, instead of nothing. So by degrees, I became familiar with the numeration table, up to a million. From 2 places of figures, I got to 3 places; - then to 4 places of figures, which took me up of course to tens of millions; then I ventured to 5 and 6 places of figures which I could eventually treat with great facility, and as already mentioned, on one occasion I went through the task of multiplying 12 places of figures by 12 figures; but it was a great and distressing effort.

Now, gentlemen, I wish particularly to impress upon you, that in order to multiply up to 3 places of figures by 3 figures, the number of facts I had to store in my mind was less, than what was requisite for the acquisition of the common multiplication table up to 12 times 12. For the latter it is necessary to retain 72 facts; whereas my multiplication, up to 10 times 10, required only 50 facts. Then I had only to recollect, in addition, the permutations among the numbers up to a million, that is to say, I had to recollect, that 100 times 100 were 10,000, 10 times l0,000 were

100,000, and that ten hundred thousand made a million. In order to do that, I had only the permutations on 6 facts, which amounted to only 18 in number, therefore all the machinery requisite to multiply up to 3 places of figures was restricted to 68 facts; whilst the ordinary multiplication table, reaching to 12 times 12, required 72 facts. Now, the importance of this is not perhaps immediately apparent, to you, but let me put an example to you. If you ask a boy abruptly "what is 900 times 80;" he hesitates and cannot answer; because the permutations are not apparent to him; but if he had the required facts as much at his command as he had any fact in the ordinary multiplication table, viz., that 10 times 10 = 100, and that 900 times 80 was nothing more than 9 times 8 by 100 times 10, he, would answer off hand 72000; and if he could answer that, he would as easily say 900 times 800 = 720,000. If the facts were stored away in his mind, so as to be available at the instant, he would give the answer without hesitation. If a, boy had that power at his command, he might at once, with an ordinary memory, proceed to compute and calculate 3 places of figures; but then there is an essential difference in the mode of manipulation, adopted by the mind, and when recording it on paper.

On paper when you multiply any number of figures, you begin with the units' places and proceed successively to the left hand, and then you add them up. That process is impracticable in the mind; I could neither remember the figures, nor could I, unless by a great effort, on a particular occasion, recollect a series of lines of figures; but in mental arithmetic you begin at the left hand extremity, and you conclude at the unit, allowing only one fact to be impressed on the mind at a time. You modify that fact every instant as the process goes on; but still the object is to have one fact, and one fact only, stored away at, any one time. Probably I had better commence with an instance or two: there are (pointing to the board) 373 by 279; I mark those two numbers down at haphazard, the result of that, is 104,067; now the way I arrive at this result is this - I multiply 200 into 300 = 60,000; then multiplying 200 into 70, gives 14,000, I then add them together, and obliterating the previous figures from my mind, carry forward 74,000; I multiply 3 by 200 = 600, and I add that on and carry forward 74,600. I then multiply 300 by 70 = 21,000, which added to 74,600, the previous result, gives 95,600, and I obliterate the first. Then multiplying 70 by 70 = 4900 and adding that amount, gives 100,500. Then multiplying 70 by 3 = 210, and adding as before, gives 100,710. I then have to multiply 9 into 300 = 2700, and pursuing the same process brings the result to 103,410; then multiplying 9 into 70 = 630, and adding again = 104,040; then multiplying 9 into 3 = 27, and adding as before, gives the product, 104,067. That is the process I go through in my mind.

Taking another example; for instance, multiplying 173 x 397, the following process is performed mentally: -

100 x 397 = 39,700
70 x 300 = 21,000 = 50,700
70 x 90 - = 6,300 = 67,000
70 x 7 - - = 490 = 67,490
3 x 300 - - - = 900 = 68,390
3 x 90 - - - = 270 = 68,660
3 x 7 - - - - = 21 = 68,681

The last result in each operation being alone registered by the memory, all the previous results being consecutively obliterated until a total product is obtained.

To show the aptitude of the mind by practice, the above process might be much abbreviated, for I should know at a glance, that

400 x 173 = 69,200

and then . . . 3 x 173 = <u>519</u>
the difference being . . . 68,681 as above.

Now, gentlemen, it must be apparent, and must be received as an established fact, that reduced to paper, mental processes do not recommend themselves as expeditious; but that, on the contrary, they are very often prolix; they are, in reality, solely designed to facilitate the registration in the mind. Although I did that sum almost instantaneously, every one of those processes was performed in my mind; but when, as you saw, I had to register them on the board, that process could not be recommended as either short, clear, or satisfactory. You will see, however, that the process I adopt is, as it were, a process of natural algebra. I have, in fact, worked out this algebraic formula $(a + b + c) \times (d + e + f) = ad + ae + af + bd + be + bf + cd + ce + cf$.

Fortunately for me I began by dealing with natural instead of artificial algebra. No man can carry any number of unmeaning symbols in his mind, but I had to deal with numbers which I understood, and I believe it was because my tuition began with this natural mode, that I attained the power I now possess; and I think it will be apparent, that teaching arithmetic in this manner, is that which is most likely to recommend itself to beginners, because you are enabled to show them, at every step, that the operation which they are called upon to execute, is that which is right in itself, and will satisfy their reason; and it has this further advantage, that unlike the ordinary mode of teaching arithmetic, which is by dogmas, the mind, instructed in the way I recommend, would have its reasoning powers generally strengthened; it would be taught to rely on itself, and thus one of the great objects of education, - that of strengthening the reasoning powers and the resources of the mind - would be generally promoted.

I now propose, after having, I hope, elucidated the process of multiplication, to show you how, step by step, I proceeded to apply the same process to other rules, even up to the extraction of square and cube roots, - compound interest, and the investigation of prime numbers. For all these questions it was necessary to invent my own rules, as I received no suggestions from any one, to assist me. All that was ever explained to me was the meaning of the square, or the cube root, or whatever was the particular branch of arithmetic to which my attention was directed; for as I said before, to show to what a limited extent my education had advanced, when I commenced seriously to calculate, my vocabulary was so restricted, that I did not even know the meaning of the word "multiply." The first time I was asked to "multiply " some small affair, say 23 by 27, I did not know what was meant; - and it was not until I was told that it meant 23 times 27 that I could comprehend the term; I believe, however, that it is not unimportant, that I should have begun without knowing the meaning of the conventional term "multiply," because the words "23 times 27" had to my comprehension a distinct meaning; which was - that 23 times 27 meant 20 times 20, plus 20 times 7, and 3 times 7 plus 20 times 3. It must be evident then, that the powers I possess are derived from careful training; which resulted very much from accident at first, and I think this want of knowledge of terms was one of the accidents, that particularly favoured my progress in arithmetic.

(In order to elucidate the position assumed, Mr. Bidder then went slowly through the process of multiplication in the mental form; recording the result at, each consecutive step and examining the previous figures during each process.)

Before proceeding to the second part of the subject, comprising the advanced rules, I must disabuse the minds of those who may have thought, that one of the objects I had in addressing this Institution was to lay a claim for myself, to some invention in the process of calculation; and that I intended to lay the foundation for myself of some credit, in detailing what appeared to me conducive hereafter to facilitating the education of youth. I beg to assure you, that such was not my intention. Indeed, perhaps, few people are less informed than myself, as to what has been written on mental calculation. With the exception of Bonnycastle, I do not know, that I ever opened a treatise on arithmetic in my life. It is therefore probable, that many of my suggestions will be found in works on arithmetic. If that be the case, I shall be much gratified, because it, will be satisfactory to me, that those minds which have been practically directed to the subject, corroborate my views as to what may be found practically useful. My only object has been to present to you a faithful record of the result of a particular training in my own mind; - to unbosom, as it were before you, its manipulations; and to leave you to draw such inferences as you may think fit.

My object has been especially to call your attention to the fact, that mental calculation depends on two faculties of the mind, in simultaneous operation, - computing and registering the result; -the faculty of computing depending on the mind having a store of facts at its command, which it may summon to its use, without apparent effort; and the latter, - the registering, - depending on the tendency of the processes to bring all calculations, as far as it may be practicable, into one result, and to have that one result alone, at a time, registered upon the mind.

I have laid great stress on the importance of beginning to study numbers and quantities naturally, before being introduced to them through the medium of symbols; and I am confirmed in this opinion, by the fact, that already, in consequence of my remarks, several gentlemen who have applied them practically, acknowledge

that the chief difficulty they have experienced, has been in retaining in their memory the figures representing the numbers with which they proposed to deal, and not the numbers themselves.

I believe that much of the facility of mental calculation, and also of mastery over numbers, depends on having the idea of numbers impressed upon the mind, without any reference to symbols. The number 763 is represented symbolically by three figures 7-6-3; but 763 is only one quantity, - one number, - one idea, and it presents itself to my mind just as the word 'hippopotamus' presents the idea of one animal. Now if you were called upon to represent the animal 'hippopotamus' by the figures 174754, it would be far more difficult to remember, because those figures have no relation to one another, - they do not guide to another sequence; and hence I feel - and it is an opinion, on which, the more I reflect, the more I am confirmed, that you should have numbers impressed on your mind as an idea connected, or identified, with themselves, and not through the 'dry-bones,' of figures. The word 'mind,' if recollected merely in connection with four symbols, or the four letters M-I-N-D, would create a much greater difficulty to the memory, than the word 'mind,' with which a signification is immediately associated.

I have already pointed out to you that, within certain limits, the power of registration keeps pace with computation; but that when such limit was passed mental computation could no longer be used with advantage. I have fixed that limit at multiplying 3 figures by 3 figures; and I do not assign that limit without reason. Each set, or series of 3 figures, constitutes a step in numbers, 787 is one series, - the second series is 787 thousand, the next series 787 millions, the next 787 thousand millions, and the next 787 billions. Therefore, at the change beyond each third figure, another idea must be seized by the mind; and though it is but one idea, yet with all the training I have had, when I pass three figures, and jump from 787 to 1787, I cannot realize to myself that it is but one idea; - in fact there are two, and this increases the strain on the registering powers of the mind.

In explaining the process of multiplication I pointed out to you the necessity of keeping only one result before the mind at a time; and you will find, throughout the whole of the remarks I shall have to submit to you, that the same plan is pursued, and that, wherever it is practicable, one result alone is presented to the memory for registration. I must impress upon you that this is the key to all other processes in arithmetic. Whoever is master of the multiplication table, and will make it his own in the way I have described, will be at no loss to find for himself a method of applying it to every other branch of arithmetic. In dealing with figures, it confers the same kind of advantage over a person who only knows numbers through symbols, as would be possessed by a man judging of the general contour of a country from an eminence, as compared with the observations of a man attempting to view it from between two hedges.

With your permission I propose now to direct your attention, shortly, to Addition, Subtraction, Multiplication, and Division, as applied to weights and measures, time, and money; to Cube and Square Roots, Compound Interest and Prime Numbers, and then to conclude with a few general remarks.

With respect to Addition and Subtraction I have little to observe, because I follow the same system as in Multiplication; beginning with the left-hand figures and proceeding consecutively to the right. By this means I have only one result to register; as I get rid of the first series of figures I have no necessity for keeping in view the numbers with which I have to deal. It does not follow that I do not recollect them; on the contrary, I invariably bear them in mind; but my object is always to relieve the mind from the feeling of oppression arising from the necessity of keeping an accurate record, and to seek for that relief by dealing with the other parts of the operation, in such manner as to accomplish it; for the only strain I have experienced, has been whenever the registering power is at all oppressed.

Division is, as in ordinary arithmetic, much more difficult than Multiplication, as it must be a tentative process, and is only carried out by a series, more or less, of guesses; but no doubt in this respect, the training arrived at by mental arithmetic gives the power of guessing, to a greater extent than is usually attained, and affords a corresponding facility in the process. Supposing for instance, that it be necessary to divide 25696 by 176, the following is the process: - 100 must be the first figure of the factor; 100 times 176 are known at once to be 17600; subtracting that from 25696 there remains 8096; it is perceived that 40 is the next number in the factor; 40 times 176 = 7040, there then remains 1056 - that, it is immediately perceived, gives a remaining factor of 6, making in all 146; thus only one result is retained in the mind at a time; but as contrasted with Multiplication it is necessary to keep registered in the mind two results which are always changing, viz.: the remainder of the number to be divided, and the numbers of the factor, as they are determined. If it is known, as is the case in the present instance, that 176 is the exact factor, without any remainder, having got the first factor - 100 - which is perceived at a glance: it is known, that there are only four numbers, which multiplied by 76 could produce a result terminating in 96, viz.: 21-46-71- and 96, and, therefore, the immediate inference is, that it must be 46, as 121 must be too little, and 171 must be too much, therefore 146 must be the factor. Thus, as before observed, the only facility afforded by mental calculation is the greater power of guessing at every step towards the result.

In multiplying pounds, shillings, and pence, - weights and measures, - or lineal dimensions, by any number, the operation must be the same; I begin with the pounds, and having multiplied them, I register them. I then multiply the shillings, reduce them into pounds, and add them on; and then the pence, and so on, until I come down to the farthings. Thus in multiplying £14 15s. 6¾d. by 787, I begin 14 times 787 = 11018; - then 787 times 15s. = £590 5s., making 11608 5s.; then we come to the 6¾d. In cases like this, various expedients frequently suggest themselves: thus, instead of multiplying by 6d., and then by ¾d., I find a facility in reducing the 6¾d. into 27 farthings, then 787 multiplied by 27 = 21249, which reduced to pounds amounts to £22 2s. 8¼d. These several sums added together give a total of £11630 7s. 8¼d.

Again, in questions involving divisions of time, distances, weight, money, &c., it is convenient to bear in mind the number of seconds in a year, inches or barleycorns in a mile, ounces and pounds in a cwt. and ton, pence and farthings in a pound sterling, &c.

Thus I worked out for myself that in a year there are 31,530,000 seconds, 525600 minutes, and 8760 hours; that in a mile there are 100,080 barleycorns, 63360 inches, 5280 feet, and 1760 yards; that in a ton there are 35840 ounces, and 2240 lbs.; and in a cwt. 1792 ounces, and 112 lbs.; and in a pound sterling, 900 farthings, and 240 pence. These, then, were so many ascertained facts, which having at my command were always ready for use, when they could be applied with advantage. Thus, for example, to find the number of seconds in 87 years. In an ordinary way you would proceed as follows: -

87 x 52 =	4524 weeks.
4524 x 7 =	31668 days.
add for odd days	87
	31755 total days.
31755 x 24 =	762,120 hours,
762,120 x 60 =	45,727,200 minutes,
and 45,727,200 x 60 =	2,743,632,000 seconds;

an operation which, worked out, involves the use of 68 figures, while I should do it in my own mind by multiplying 31,536,000 by 87, requiring by the longest process only 26 figures, and yet giving the same result. As an example in lineal measure, suppose it is required to find the number of barleycorns in 587 miles, the ordinary process, viz., - 1,760 x 587 x 3 x 12 x 3 = 111,576,960, when worked out, requires 50 figures; while, mentally, I should multiply 190,080, the number of barleycorns in a mile, by 587, which would not require half the number of figures. These instances will render sufficiently evident, the great facility that is given, even in ordinary arithmetic, by having at command such a store of facts, as those to which I have alluded.

I now come to the question of Square and Cube Roots. Nothing ever excited so much surprise on the part of those who examined me, as the facility with which questions were answered in those arithmetical rules. Yet there is no part of mental calculation for which I am entitled to less credit. In fact, it is a mere sleight of art, as I shall show you. When I was first asked to extract the square root, I did not know what the term square root meant, and this was explained by saying, as 400 = 20 x 20, that 20 was called the square root of 400; and similarly that as 8 = 2 x 2 x 2, that 8 was the cube of 2, and 2 was the cube root of 8.

Having received this explanation, I devised my own rules for performing the operation, and this was facilitated by the fact, that the numbers submitted to me were almost invariably perfect, squares, or cubes, arising from the circumstance that, in order to gave themselves trouble, those

who questioned me, squared, or cubed a number, as the readiest mode of testing the accuracy of my reply, which being found correct, they were satisfied and so was I. The consequence was, that nearly every example proposed was a true square, or cube; hence I hit upon, the following expedient. It appeared on reflection, that whatever might be the two last figures of a true square, as for instance 61, it could only be produced by the square of four numbers, viz., 19-31-69-81; hence if called upon to extract the square root of 337,561, I saw as easily as I saw that 5 was the nearest square root to 33, that 500 was the nearest square root to 330,000, and consequently, that 581 was the square root, inasmuch as 81 stands nearly in the same relation between 500 and 600, as 337,561 does between 250,000 and 360,000, the squares of 500 and 600 respectively.
In reference to square numbers terminating in 25; although all numbers ending in 5, when squared, give 25 as terminals, I noticed that the squares of numbers ending in 5, 45, 55, and 95, ended in 025, and that those of the numbers terminating, in 15, 35, 65, and 85, ended in 225; whereas the squares of those ending in 25 and 75, ended in 625. Hence in extracting the square root of 442225, I perceived, as before, that 600 must be the first factor, and that the last one must lie between 15, 35, 65, and 85, and judging from the position of 442,225 between 360,000 and 490,000, the squares of 600 and 700 respectively, I saw that 65 was the factor required, and the root therefore was 665.

Now, as to the Cube Root, it was still more simple, because there were fewer numbers to select from: thus if it were an even number terminating in 76; there were only two numbers, 26 and 76, of which the cube would terminate in 76. If the number was an odd one, terminating, for example, in 17, and it was a true cube : then the root of that number must terminate in 73, as no other number, when cubed, would terminate in 17. Now, as regarded numbers terminating in 5; the cubes of some of them, namely, 5, 25, 45, 65 and 85, terminated in 25; while those of 15, 35, 55, 75, and 95, terminated in 75.

For example, if called upon to extract the cube root of 188,132,517, I knew, that as the cube of 5 was 125 and that of 6 was 216, the first figure in the root must be 500; and as 73 was the only number left, which, being cubed, could produce a sum ending in 17, I guessed, with accuracy, that the cube root was 573. So I proceeded similarly in finding the cube root of 180,362,125, but I was then obliged to take a little more time than in the other case. I knew that 5 was the first figure, but whether the two last figures should be 65 or 85 I was not certain: I cubed 560, and found it was 175,616,000, and then I perceived that the root must be 565.

I used thus to arrive at a square, or cube root, in my own way; but if I suspected that the number was not a perfect square, or cube, I tested it by 'casting out the nines' - a process familiar to arithmeticians; and in such cases, the results were approximated to by a tentative process, and no advantage whatever could be derived from the methods here described.

This leads me to the subject of Compound Interest, relative to which I am afraid I shall find some difficulty in making myself understood as clearly as I should desire. I am especially anxious as to this rule, because if I succeed in explaining it satisfactorily, you will comprehend the mode of reasoning by which I investigated rules and ascertained some of the properties of numbers, and more particularly of a series of numbers. I need not tell you that calculating Compound Interest, without logarithms, is a matter involving immense labour, especially if it extends over a great number of years. It is raising a fractional number to a high power, which thus becomes an expanded series and no terms can be neglected, because the ultimate value of any one term cannot he foreseen. When called upon to perform sums in Compound Interest, I found them such a source of labour and of strain upon the memory, that I was induced to seek for some means of relieving it from the pressure, and in doing this, I adopted a mode of reasoning which I will try to explain; as from it you will perceive the tentative process, by which I arrived at all the rules I adopted in arithmetic.

I will assume, that I was called upon to calculate the compound interest for £100 at 5 per cent, for 14 years. In the ordinary way, without the use of logarithms, you would begin thus, -

100 x (1.05) x (1.05) x &c. .... 14 times.

Now 1.05 x 1.05 = 1.1025, and 1.1025 x 1.05 = 1.157625;

which process, carried only a very little further, will involve so many figures, as to be quite impracticable for the memory.

Now the way in which I proceeded was this; I considered that if £100 was put out at simple interest for 14 years it would produce 14 x £5 = £70. I then regarded each £5 put out at simple interest; thus the first £5 will produce 5s. per aunum, but as it commences at the end of the first year it will be for a period of 13 years instead of 14 years. In the second year the £100 gives birth to a second £5 which carries simple interest of 5s. per annum for 12 years. Similarly the third £5 carries interest for 11 years, and so on, the number of years regularly diminishing, hence the total amount of interest at 5s. per annum is expressed by a series -

5s. x {13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1}

I had therefore presented to me the problem to sum that series.

It so happened that before I was called upon to deal with compound interest, I was asked the well-known question, "If a man pick up 100 stones a yard apart, picking them up separately and putting them in a basket, how far must he travel?" That question set my mind thinking over the series which is embodied in its solution, and the method I adopted for obtaining the summation is as follows: - Writing the series the other way we have 1 + 2 + 3 + 4 + 5+ 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13; now add the successive terms together, one after another 1 = 1, 1 + 2 = 3, 1 + 2 + 3 = 6, 1 + 2 + 3 + 4 = 10, and so on, and write them, so as to form another series, viz., 1, 3, 6, 10, 15, &c.

Divide each of these terms by the numbers representing their order as

	1st	2nd	3rd	4th	5th	&c.
viz.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{6}{3}$	$\frac{10}{4}$	$\frac{15}{5}$	&c.
or	1	1½	2	2½	3	&c.

in which it will he seen, that these quotients form a regular progression, each term of which is half an unit more than half the number expressing the position of the term in the series;

for example, in the 4th term, the corresponding quotient is 2½ or ½, more than 4/2 and so on; hence if 13 he the number of terms as in the case under consideration) we obtain the sum of 13 terms = 13 x (13/2 + ½) = (13 x 14) / 2 * and therefore the interest = ((13 x 14) / 2) x 5s. = £22 15s. 0d. This added to the £70 before mentioned, gives a total of £92 15s. 0d.

The next stage of the proceeding was to consider each 5s. as put out at simple interest, namely 3d. per annum.

Now the first 5s. (the interest upon the first J5) accrues in the second year, and therefore will carry simple interest for 12 years. The second year's simple interest of 5s. arising out of the first £5 will similarly run for 11 years and so on.

Thus it appears that the calculation of the simple interest upon 5s., the interest upon the first £5, is similar to that of the 5s. the interest upon the first £5, but having one year less to run, namely, 12 instead of 13 years; hence it is expressed by the formula

$$\frac{n\,(n+1)}{2} \times 3d,$$

In the same way the simple interest upon the 5s., - the interest upon the second £5, - is the result of the summation of a series like the last, except that it was for 11 years; and so of the rest Therefore the series expressing the total amount of interest upon all the 5s. has its terms made up of the sums of corresponding terms of the former series, or 1 + 3 + 6 + 10 + 15 + 21, &c., giving rise to 1, 4, 10, 20, 35, &c., in which we have 1 = 1, 1 + 3 = 4, 1 + 3 + 6 = 10, 1 + 3 + 6 + 10 = 20, and so on.

Now to sum this series, divide each term by the corresponding term in the previous series thus -

$$\frac{1}{1} \quad \frac{4}{3} \quad \frac{10}{6} \quad \frac{20}{10} \quad \frac{35}{15} \quad \frac{56}{21}$$

giving quotients

$$1 \quad 1\tfrac{1}{3} \quad 1\tfrac{2}{3} \quad 2 \quad 2\tfrac{1}{3} \quad 2\tfrac{2}{3}$$

$$\text{or} \quad \frac{3}{3} \quad \frac{4}{3} \quad \frac{5}{3} \quad \frac{6}{3} \quad \frac{7}{3} \quad \frac{8}{3}$$

which, as in the former case, are in regular progression, and we observe that in any term, the fifth for example,

$$35 = 15 \times 2\,\frac{1}{3} = 15 \times \frac{7}{3} = 15 \times \frac{5+2}{3}$$

* Hence if n be the number of terms, we have the sum of n terms =

$$\frac{n\,(n+1)}{2}$$

for the general formula.

or the numerator of the fractional multiplier, is the number of the term of the series + 2, and so of the rest.

But the 15 is, as we have seen, made up of the summation of 5 terms of the previous series (1 + 2 + 3, &c.) and hence may be represented by

$$15 = \frac{5 \times (5+1)}{2},$$

$$\text{and the 35 by } 35 = \frac{5 \times}{} \frac{(5+1)}{2} \frac{(5+2)}{3}.$$

Now observe, this interest accrued in the second year, hence to refer the terms of the last series to the principal epoch, whence the 14 years is estimated, we have only to put 7 - 2 for 5 and we obtain

$$35 = \frac{(7-2)}{3} \frac{(7-1)}{2} \frac{7}{} = \frac{7}{1} \frac{\times\,(7-1)}{2} \frac{(7-2)}{3}$$

and, therefore, if a = number of years = 14, in this case we get the expression

$$\frac{a}{1} \frac{(a-1)}{2} \frac{(a-2)}{3} \times 3d,$$

which put in figures gives

$$\frac{14 \times 13 \times 12}{1 \quad 2 \quad 3} \text{ x 3d} = \text{£4 11.s. Od},$$

and this added to the previous sum of £92 15s. Od produces £97 6s. 0d.
I then considered each 3d. put out at simple interest, and pursuing the same mode of investigation, I found the summation of the series represented by ,

$$\frac{a}{1}\frac{(a-1)}{2}\frac{(a-2)}{3}\frac{(a-3)}{4},$$

and the interest to be

$$\frac{a}{1}\frac{(a-1)}{2}\frac{(a-2)}{3}\frac{(a-3)}{4} \text{ x } \frac{3d.}{20} = \text{12s. 6d.;}$$

this added to £97 6s. 0d. gives £97 18s. 6d.
Similarly the next interest is expressed by the formula

$$\frac{a}{1}\frac{(a-1)}{2}\frac{(a-2)}{3}\frac{(a-3)}{4}\frac{(a-4)}{5} \text{ x } \frac{3d.}{20 \text{ x } 20} = \text{1s. 3d.,}$$

which added gives £97 19s. 9d., and the next

$$\frac{a}{1}\frac{(a-1)}{2}\frac{(a-2)}{3}\frac{(a-3)}{4}\frac{(a-4)}{5}\frac{(a-5)}{6} \text{ x } \frac{3d.}{20 \text{ x } 20 \text{ x } 20} =$$

$$= \frac{14}{1}\frac{13}{2}\frac{12}{3}\frac{11}{4}\frac{10}{5}\frac{9}{6} \text{ x } \frac{3d.}{20 \text{ x } 20 \text{ x } 20} = \text{1d.,}$$

which added gives £97 19s. 10d.

Perceiving how rapidly the series converged, and that the remaining terms could not possibly amount to one farthing, the process was stopped and the result stated as above, £97 19s. l0d, which, instead of requiring fourteen operations was arrived at in live, and these of much easier computation than may probably appear from the description just given.

Before quitting this part of the subject I will venture to express the total compound interest of any sum of money algebraically, in another more simple form.

Let P represent the principal sum,
r the rate of interest (= 1/20, if 5 per cent, for example),
n the number of years.

Then compound interest =

$$\text{P X } \{nr + \frac{n}{1}\frac{(n-1)}{2} r^2 + \frac{n}{1}\frac{(n-1)}{2}\frac{(n-2)}{3} r^3 + \ldots + \frac{n}{1}\frac{(n-1)}{2}\frac{(n-2)}{3}\frac{\ldots}{\ldots}\frac{(n-(n-1))}{n} r^n\},$$

and the amount, or interest and principal together, =

$$\text{P X } \{1 + nr + \frac{n}{1}\frac{(n-1)}{2} r^2 + \frac{n}{1}\frac{(n-1)}{2}\frac{(n-2)}{3} r^3 + \ldots + \frac{n}{1}\frac{(n-1)}{2}\frac{(n-2)}{3}\frac{\ldots}{\ldots}\frac{(n-(n-1))}{n} r^n\},$$

which is the expansion, by the binomial theorem, of the expression P $(1 + r)^n$, which is the form in which the problem is presented for solution by logarithms, or by successive involution.

I do not in any way claim the discovery of a new principle; but I have endeavoured to show by what kind of process my mind, at n very early age, and when wholly unacquainted with symbolical representation and algebraic expedients, analysed the law connecting these series and rendered them available for computation.

As an example of how this was carried out mentally, I will take £100 as put out at 2 per cent, for 35 years. Then

the first term = P n r =		$\frac{100 \times 35}{50}$	=	£70;
the second	=	$\frac{70 \times 34}{2 \times 50}$	=	£23 16s. 0d., which added to the foregoing gives £93 16s. 0d.;
the third	=	$\frac{\text{£23 16s.} \times 33}{3 \times 50}$	=	£5 4s. 8½d., giving £99 0s. 8½d.;
the fourth	=	$\frac{\text{£5 4s. 8½d.} \times 32}{4 \times 50}$	=	16s. 9d., giving £99 17s. 5½d.;
the fifth	=	$\frac{\text{16s. 8d.} \times 31}{5 \times 50}$	=	2s. 1d., giving £99 19s. 6½d.;
the sixth	=	$\frac{\text{2s. 1d.} \times 30}{6 \times 50}$	=	2½d., giving £99 19s. 9d.;
the seventh	=	$\frac{\text{2½d.} \times 29}{7 \times 50}$	=	¼d., giving £99 19s. 9¼d.;

which of course is the, last, the remaining terms obviously giving very small fractions. Mentally, however, this would not, occupy me more than a, minute, so that the result is arrived at almost as quickly as by logarithms.

After Compound Interest the next subject is that of Prime Numbers, which it is hardly necessary to say, are those which are only divisible, by themselves and unity, and having no other known relation to any number below them. Even at, the present time I cannot help occupying myself with what I feel to be impossible in calculation, and for years I studied these numbers in the vain hope of discovering some law of sequence, which they obey. On thinking over this matter (and in attempting an analysis of any kind, though you may not succeed, you generally find out something which proves to be more, or less useful) my speculations led to the discovery of processes by which great facilities were offered for ascertaining the factors of a number, and thereby determining whether it was a prime, or not.

When required to ascertain the factors of any proposed number, the obvious mode, if it be even, is to divide by 2 successively until an odd number be arrived at. That done, I add the digits up and if divisible by 3, I divide by 3 till it is exhausted. Having got to that step, all trace is lost of the components and nothing is left but to commence dividing it by the consecutive prime numbers 7, 11, 13, 17, 19, 23, 29, &c., up to its square root. If none of these numbers will divide it, it may be inferred that it is a prime number. This of course is an exceedingly laborious and tedious process, and I had a great desire to discover some way of abbreviating it. The first thing that occurred to me was this - suppose an odd number to have two factors, they must both be odd, and being odd can be expressed in whole numbers by a + b for one and a - b for the other, and the number itself by $(a + b)(a - b) = a^2 - b^2$ from which it is clear, that no prime number can represent the difference between two squares, for if so it can be divided by (a + b) and (a - b), with, however, one exception, which is when a - b is unity; thus 7 is the difference between the square of 4 and the square of 3; 19 is the difference between the squares of 10 and 11; and 23 is the difference between the, squares of 12 and 11; but no number can be resolved into factors, unless it be the difference between two true squares.

Now this afforded a rapid test whether the number was divisible by the powers nearest to the approximate square root, and which, being the largest, were the most difficult to divide by. For example, suppose it, be required to ascertain the factors, if any, of the number 3139; the nearest square exceeding this is 3249, the root of which is 57, the difference 3249 and 3139 is 110, which is not a true square; but take the next square above, that is 3364, the root of which is 58 or 57 + 1, the square of this exceeds that of 57 by 57 x 2 + 1 = 57 + 58 = 115, which added to 110 gives 225, which is a true square the root of which is 15, hence

$$\begin{aligned} 3139 &= 58^2 - 15^2 \\ &= (58 + 15)(58 - 15) \\ &= 73 \times 43 \end{aligned}$$

By this it will be seen, that the necessity is avoided of dividing by 53, 47, 43, the primes between the square root as above and the actual factor, and it is an easy process, because having begun with 110, the next square gives an addition of 115, the next 117, the next 119, and so on, the addition of 2 in each case corresponding with an additional unit in the square numbers 57, 58, 59, &c., but it will occur to you that there is a certain point where you get wide away from the square root, and you have to go through a great number of additions, before you obtain a similar amount in the divisor. When you arrive at that point, this process should be abandoned, and you must fall back on dividing by 7, 11, 13, &c., and so on till the quotients are ascertained.
But even here the process can be abbreviated, and I do it in this manner.

Suppose I want to know if 23141 be divisible by 79. Instead of beginning at the left hand I begin at the opposite, or unit end, and the square of 79 being 6241, which deducted from 23141 leaves 16900, if this be divisible by 79, then 169 must be so also, which I see at a glance is not the case, and therefore I go on to another. I do not want to divide until I get a remainder; all I want is to ascertain if it be divisible, or not.

I try it therefore by 73; which must be multiplied by 17 to produce a number ending in 41, that is 1,241, which deducted from 23141 gives 21900, which is divisible by 73 without remainder. Therefore the factors are 73 and 317.

There is still another expedient for ascertaining the factors of a number, founded upon the principle familiarly called the property of the nines. Suppose you have two numbers 713 and 328, and you add up the digits and divide them respectively by 9, thus: -

$$\frac{7+1+3}{9} = \frac{11}{9} = 1 \text{ and remainder } 2,$$

$$\frac{3+2+8}{9} = \frac{13}{9} = 1 \text{ and remainder } 4,$$

that you multiply these numbers together, thus, 713 x 328 = 233864, and add up the digits and cast out the nines, thus, (2 + 3 + 3 + 8 + 6 + 4)/9 = 26/9 = 2, with remainder 8; then this remainder 8 = 2 x 4, the product of the remainders arising from

the former operations, or the remainder after dividing that product by 9, as in the following example: -

764 and 329;

$$7 + 6 + 4 = 17, \text{ and } \frac{17}{9} = 1 \text{ and 8 over;}$$

$$3 + 2 + 9 = 14, \text{ and } \frac{14}{9} = 1 \text{ and 5 over;}$$

But 764 x 329 = 251,356, and 2 + 5 + 1 + 3 + 5 + 6 = 22,

$$\text{and } \frac{22}{9} = 2 \text{ and 4 over;}$$

$$\text{also } \frac{8 \times 5}{9} = 4 \text{ and 4 over.}$$

Now to apply this to discover the factors of a number: take 23,141, as in the former example, in which the sum of the digits, viz., 2 + 3 + 1 + 4 + 1, = 11, and 11 / 9 = 1 with remainder 2.
Let us inquire whether 71 will divide 23,141, and first since 7 + 1 = 8, if 71 be a factor, the other factor must be such that the sum of its digits, divided by 9, must give a remainder of 7; because 8 x 7 = 56, and 56 / 9 = 6 and remainder 2 as above.

Now, the only number which, multiplied by 71, will give 41 as terminal figures is 71; but, as already observed, the digits of the other factor, if it exists, must give, when divided by 9, a remainder of 7; the factor, therefore, must be 871, in which 8 + 7 + 1, divided by 9, gives a remainder 7; however, a mere glance shows 871 to be far too large, hence 71 cannot be a factor of 23141.

I next try 73, in which 7 + 3 = 10, gives, after division by 9, a remainder of 1, therefore, if this be a factor, the other factor must be such a number that the sum of its digits, after division by 9, must give a remainder of 2, or 9 + 2, or 18 + 2, &c., because 1 x 2 = 2, the remainder after dividing the sum of the digits of 23141 by 9. But as 17 is the only number which multiplied by 73 will give a product terminating in 41, the factor, if it exists, of which 17 forms the two terminal digits, requires another digit, to make the sum of its digits = 9 + 2 = 11. This then is 3, and 3 + 1 + 7 = 11, and as this number 317 obviously may be a factor, I multiply 317 x 73 and find it produces 23141.

These are some of the expedients by which the tedium and labour of investigating prime numbers may be relieved, but, except as a matter of curiosity, I never found much use in pursuing them further.

Before concluding, I will, with your permission, make a few remarks on the mode by winch the faculty of calculation can, in my opinion, be beneficially applied in our own profession. Now, to do this it is requisite especially that, you should have a clear idea of the facts required by mental arithmetic, - of some of the facts connected with the laws of mechanics, - and considerable experience in the practice of the profession. The power of mental calculation has undoubtedly been of the highest importance to me, and has compensated for many defects in my professional education. That power was, moreover, the means of introducing me, at an early period, into a particular branch of our profession, - that of Parliamentary Inquiries, - and nearly the first occasion on which I was engaged before a Committee of the House, was on the original Manchester and Liverpool Bill. I was then an Assistant to the late Mr. Henry Robinson Palmer, who was engaged for the Canal interest in opposing the Railway. The late Mr. Adams, who was connected with the Duke of Bedford, Mr. Serjeant Spankie and Mr. Brougham, were Counsel for the Bill, and the present Mr. Baron Alderson was in opposition. I contrived, on that occasion, to make myself very disagreeable in suggesting matter for cross-examination, and a curious incident took place. When the Bill arrived in Committee in the House of Lords, Mr. Adams suggested to their Lordships that I should not be allowed to remain in the room, because "nature," he said, "had endowed me with particular qualities that did not place my opponents on a fair footing with me." That suggestion was not entertained by their Lordships, and I was allowed to indulge my "particular qualities," throughout the whole of that inquiry, although my opposition was, fortunately for all of us, utterly unsuccessful, in spite of my best efforts. The Bill I was opposing was carried, and I "believe it was, in a great degree, owing to the success of that Bill, that we are now enabled to meet together in such numbers in this Institution.

There is another instance in which I applied my knowledge with some success, and it occasioned some amusement, us will be recollected by several friends, here, who were present on the

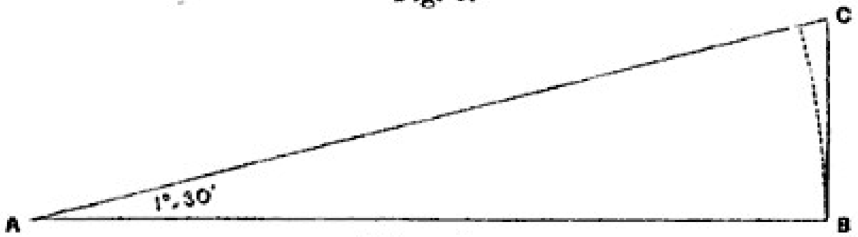
occasion. The Northampton and Peterborough Railway Bill was violently opposed by a compact body of landowners. Among other objections to the Bill, the effect of the works on the floods, in that valley, was particularly insisted on. The great objection was, not so much that we should embay the flood waters, while they were in a state of motion, but that when the flood abated, the water would be retained on the land, so long as to chill and injure the, vegetation. As you well know, learned Counsel when they think you cannot give them a definite answer adopt a particular style of cross-examination, somewhat to this effect; they ask, "Will you undertake to say that the flood water will not be retained for a week?" and when the witness replies, "I do not think so;" they will say, "Will you undertake to say it will not remain there for a fortnight, or three weeks ?" and so they go on, till they make out that your opinion is worth nothing. Mr. Sergeant Kinglake asked me such a question, and I determined I would give him a *bona fide* answer. I took the Ordnance map, and measured the area of the land in question, taking care to obtain a full area, so that he could not trip up my reply in that direction. I forgot the exact figures, but assuming it be 200 acres, I said, "I will take 9 million of feet as the area of 200 acres (which is also an excess), I will assume, that the depth of the flood is 3 feet (which is also an outside quantity), that makes 27 millions of feet upheld on this land." The question was, "How rapidly will that pass through the bridge?" The bridge had, say 1,000 feet of water-way, the head I insisted on, which would be the maximum head occasioned by the retention by the piers of the bridge, was 3 inches; the theoretical velocity due to that was 4 feet, and the practical velocity was 2½ feet; this gave 2,500 cubic feet per second; I therefore immediately answered, that "the flood would be upheld for three hours!" Now I was perfectly well aware the learned Counsel could not test the process, but at the same time I was determined not to leave anybody the means of picking a hole in my garment; so I gave a bona fide result, to the manifest discomfiture of my friend of the long robe: and another learned Counsel, with the usual amiable desire of helping 'a lame dog over a stile,' suggested, (when he saw how they were taken aback,) that, perhaps, his learned Brother might get an equally satisfactory answer if he asked me, how many fish went through in the same time? An instance of practical utility would be this: A fact to be registered in the mind is, that approximately 220 cubic feet of water will flow, per minute, through a pipe of 12 inches diameter, at an inclination of 1 in 100. Now from this fact there is no difficulty in ascertaining, approximately, the quantity of water flowing through any sized pipe, at any inclination. Assuming the inclination of a pipe of 3 inches diameter to be 1 in 400, the general laws of hydraulics demonstrate that the velocity will be reduced one-half by the lesser inclination, and one-half by the lesser diameter, the two combined reducing it to one-fourth in velocity; and the diameter of 3 inches as compared to 12 inches, giving an area of one-sixteenth, the result would be that the flow would he reduced (approximately) to one-sixty-fourth, or be about 3½ feet per minute.

I cannot recommend, that, the Engineer should rely implicitly upon this mode, for ascertaining the details of his plans, or designs, but it will enable him, in times of leisure, to arrive at the outline of his plan, in the same manner that the Artist prepares the rough sketch of his picture, and subsequently fills in the details.

An incident which occurred lately illustrates forcibly the way in which mental arithmetic may be made to afford rapid and valuable approximative results, without having recourse to complicated formulae or to books of tables, in a case to which they appear peculiarly applicable, and thereby suggesting methods of verification and of experimenting in questions of considerable practical importance.

I recently accompanied our President to Manchester, where we witnessed some interesting experiments in gunnery, then being made by our Member, Mr. Joseph Whitworth. In one of the trials a gun was laid at an inclination of 1 degree 30 minutes, in a gallery 500 yards in length; on firing, the balls were observed to pitch on a level with the points from which they started.
It occurred to me to endeavour to ascertain the velocity of the shot. I did it by this process: -

Fig. 1.



A being the point from which the shot started and **B** the spot where it pitched; then after leaving **A** in the direction **A C**, at an angle of 1 degree 30 minutes to the horizon, if uninfluenced by gravity, it would after traversing 500 yards reach the point **C**.

From this mode of considering it, we may calculate the distance **C B**, for **A B** being the radius of a circle = 500 yards, we have the entire circumference = 3140 yards nearly, and since 1°30' is the 1/240 part of 360, we have 3140/240 = 13 yards 3 inches = 39,25 feet for the length of **C B**.
Now the effect of gravity, when acting, is to make the shot fall from **C** to **B**, or through this height of 39,25 feet, and the time occupied will be

$$\frac{\sqrt{39{,}25}}{4} = \frac{6{,}25}{4} = 1\ \frac{9}{16}$$

seconds, but as sound travels at the rate of 1140 feet per second, the noise of the shot striking the far end would take $1^3/_8$ seconds in traversing the 500 yards to reach the point whence the gun was fired and where we were standing, and therefore the whole interval between the firing the shot, and its fall being heard would be about 3 seconds. This result could therefore be tested, and when tried by the President with a stop-watch was proved to be quite correct; and hence the mean velocity of flight was 1500 / $1^9/_{16}$ feet = 960 feet per second.
It now only remains for me to lay before you the mode by which, I think, mental arithmetic should be taught; though you will doubtless, to some extent, have already formed a general idea, from my remarks, what that method should be.

I think it most essential that, numbers should be taught before figures - that, is to say - before their symbols and probably even before the letters of the alphabet are learnt. The first step should be to teach the child lo count up to 10 and then to 100. He should then be instructed to form his own multiplication table, by connecting rectangular pieces of wood, shot, or marbles, or any symmetrical figures: probably marbles may be the best, as they are the very early associates of the child, and may be considered in some degree as his playmates, and will therefore be likely to form the most agreeable associations in his mind. Having formed these rectangles, he will be enabled by his previous experience in counting, to reckon the number of pieces in any rectangle, and thus to demonstrate to himself all the facts of the multiplication table, up to 10 times 10. Having thus acquired the multiplication table up to 100, he should then be taught to count up to 1000 by 10's and 100's. It would not then be difficult to teach him to enlarge his own multiplication table. In the first, place he would have no difficulty in multiplying 10 by 17, because he will be quite familiar with the fact that 10 times 10 are 100, and 10 times 7 are 70, and adding them together will give the result, represented by 170. Jt will then be easy to follow this by multiplying 17 by 13. He knows already that 10 x 17 is 170 and that 3 times 10 are 30, which added gives 200, and that 3 times 7 are 21, which added gives 221, the result required. Hy patience and constant practice in this way, he would gradually be taught to multiply 2 figures by 2, and eventually 3 figures by 3. After this he will be led upon the same principles to the application of his faculties to the other rules of arithmetic.

But, I would suggest that this mode of proceeding presents advantages of much greater importance than even the teaching of figures; for far beyond the mere facilities in computation, would be the advantages afforded by the opportunity of making this branch of education conducive to the highest objects to which education can be directed; that is, to the cultivation of the reasoning powers in general.

I would therefore introduce a boy, through this means, to natural geometry and algebra. By placing shots, or any small symmetrical objects on the circumference and the diameter of a circle, he would he able, by actual observation, to satisfy himself of their relative proportions. He might simultaneously be taught the relation of the area of the circle to the area of the square. He might also be taught the beautiful problem, that the square of the hypotenuse equal the squares on the other sides of a right-angled triangle - that the areas of all triangles on the same, or equal bases, and between two parallel lines, are equal. Of these, and many other useful facts, he would satisfy himself, long before he could appreciate the methods by which they are demonstrated in the elementary works on mathematics. Advantage may also be taken of this mode to develop many other ideas connected with geometry, as, for instance, that all the angles subtended from the same chord in the circle are equal. This might be shown by having a small angle cut in pasteboard, and fitted to every possible position in which two lines could be drawn within the circle upon the same chord. He might also be taught that the rectangles of the portions of any two lines intersecting a circle are equal. On this point. I need not enlarge, because to all of you it will be apparent, that many other useful properties might be thus imprinted on the youthful mind. So again, as regards the series I have mentioned - the stones in the basket, for instance, and also the summation of the scries 1+3+5+7+9: the summation of this series is equal to the square of the number of terms required to be summed up. If the learner once acquired a feeling for the beauty of the properties of figures - surmising that be had any natural taste for arithmetic - the discovery of these facts by his own efforts might incite him to farther investigations, and enable him to trace out his own path in the science.

I would again, however, observe, that I should despair of any great, success in the pupil's progress in the science of arithmetic if he did not commence before he knew anything of symbols, and if his first conception of numbers was not derived from their real tangible quantity and significance.

I fear that the remarks I have made may not have been quite so clear as might have been desirable; but if they should lead to any practical, or useful result, I shall be amply repaid for the exertion which I admit it has cost me, in making the attempt to elucidate the subject; and if I have been at all successful I am bound to confess that it will be chiefly owing to the kind, generous, and sympathising support with which you have been pleased to encourage me.

Before, however, I cease to occupy your attention, I have one observation to make, of a personal nature. You may think that I attach much more importance to the science of arithmetic than it is justly entitled to. If, however, that be so, you must at the same time admit that I have reason for this feeling. It is to that science that I am indebted for the friends who encouraged my early career, as well as for those who afforded me the benefits of the comparatively small amount of education I received, - it procured me the friends, who opened to me, in manhood, the patlis in the profession to which it is my pride to belong; - more than all, it has secured me the inestimable friend who now presides over us, and giving me confidence and the power of analysis, it has materially aided in raising me to the position I now hold. Every scene, however sunny, has its shadow, and I fear that though I have had strong, and steadfast friends, still, probably in consequence of some infirmity of temper, and the habit of early independence of thought and of self-reliance, I have not at all times studied, with the care I ought to have done, the feelings of my Brethren in the Profession, and thus I have not attained with all of them, the popularity I could desire. I hope, however, that my career will not close, before a better feeling is established with all of them; and especially with those who are associated with me in tliis institution, of which I have been a Member, for upwards of thirty years.

Mr. R. STEPHENSON, PRESIDENT:- "I need scarcely ask you, whether a vote of thanks shall be offered to Mr. Bidder, for the very interesting exposition he has given of the system of Mental Calculation, which he may be said to have invented, and which he uses with such remarkable facility. You must all have been as much struck with the simplicity of the processes, as with the clearness of the modes of reasoning by which those processes were arrived at, and the rapidity of the application of the system, but you are not aware of the labour it has cost our friend to prepare these lectures, and nothing but his friendship for me, and anxiety to signalize my Presidency, by what must be an unique production, would have induced him to undertake the task. His proposition for teaching to youth a system of mental arithmetic is well-deserving of serious consideration; but whatever training may be essayed, I fear that few Bidders will result; it will be rare to find the large capacity and pure analytical mind which characterizes our friend, and without which the process we have heard explained would be but of little use. I cannot expect that any discussion will ensue upon a communication of this description, and therefore I would at once adjourn the meeting; but I must before doing so notice the very graceful manner in which our friend has alluded to self assumed faults, which we all know are more, I may say, assumed than real, as all who have the advantage of intimacy with him in private life, must fully testify; and in his public career there are many noble and kind acts of his, deserving of record. In truth, Gentlemen, we may all mutually ask a general amnesty for acts of professional rivalry; and we should scarcely be in the position we are, but for this institution, which first softened those feelings of professional jealousy which kept our predecessors apart and it is to our meetings here that may be attributed, in a great degree, the kindly feelings we entertain for, and the mutual assistance we render to each other. Let us then cherish these feelings, and endeavour with all our energies to support the Institution which has been, and will be, productive of such good results, and by no means can this be done so effectually as by following the example of our friend Mr. Bidder, in presenting good and interesting papers, such as that for which I now propose a cordial vote of thanks."